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Application of Variational Homotopy Perturbation Method for the Korteweg-de Vries Equation

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Abstract- In this paper, the Korteweg-de Vries Equation which incorporates convection and diffusion in fluid dynamics, and to describe the structure of shock waves is solved by Variational Homotopy Perturbation Method. The numerical solutions obtained are compared with the exact solution which demonstrates the accuracy and efficiency of the method. The results reveal the applicability of the method to other nonlinear problems.

Keywords-Variational Homotopy Perturbation Method (VHPM); Korteweg-de Vries (KdV) Equation; Solitons.

1.INTRODUCTION

In 1984, while conducting experiments to verify the most competent design for canal boats in the Union Canal at Hermiston, nearby to the Riccarton campus of Heriot-Watt University, Edinburgh ,a Scottish engineer named John Scott Russell was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped it left behind solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. Russell named that phenomenon as Wave of Translation. Further investigations were done by Airy, Stokes, Boussinesq to comprehend this phenomenon.In 1895, Korteweg and Vries after performing a Galilean and variety of scaling transformations derived KdV equation to model Russell's phenomenon of solitons [1]

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He saw the solitary wave as a independent dynamic unit, a thing which possess many properties of a particle. From the modern standpoint it is used as a constructive element to formulate the complex dynamical behavior of wave systems all the way through science: from hydrodynamics to nonlinear optics, from plasmas to shockwaves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought.

It is motivating to explore the beauty of the Korteweg-de Vries Equation (KdV) which is nonlinear which exhibits special solutions, known as solitons or solitary waves. Solitons are stable and do not disperse with time. Furthermore there are solutions with more than one soliton which can move towards each other, interact and then emerge at the same speed with no change in shape. The Korteweg-de Vries is a hyperbolic PDE from that it follows that it describes a reversible dynamical process.KDV has motivated considerable research into analytical and numerical solution by several methods. Recently the study of solitons has been the focus of many researchers. Ganji DD and Heidari M [2] solved this equation by Homotopy Perturbation method. Wazwaz AM [3] obtained rational solutions by ADM. Wang C et al [4] applied HAM to solve this equation, whereas He JH and Wu XH [5]; He JH [6] applied both VIM and HPM to obtain solitary solution. Many authors Gardner CS et al [7], Khattak AJ and Siraj-ul-Islam [8] solved KdV equation numerically.

Here, the Korteweg-de Vries equation is solved by Variational Homotopy Perturbation Method (VHPM)[9,10] which converges very fast to the results. Moreover, contrary to the conservative methods which require the initial and boundary conditions, the VHPM provide an analytical solution by using only the initial conditions. The solution is presented graphically by Mathematica.

2.VARIATIONAL HOMOTOPY PERTURBATION METHOD

To convey the basic idea of the Variational homotopy perturbation method, we consider the following general differential equation

$$Lu + Nu = g(x) \qquad (2.1)$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the forcing term. According to variational iteration method, we can construct a correct functional [6] as follows:

$$u_{n+1}(x) = u_{n(x)} + \int_0^t \lambda(\tau)\left(Lu_n(\tau) + Nu_n(\tau) - g(\tau)\right)d\tau \qquad (2.2)$$

Where the general Lagrange multiplier is λ, which can be identified optimally via variational theory. We apply restricted variations to nonlinear term Nu so that we can determine the multiplier.

The Variational Homotopy Perturbation Method [11] is obtained by the elegant coupling of correction functional of Variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_{0(x)} + p\int_0^t \lambda(\tau)\left(\begin{array}{l}\sum_{n=0}^{\infty} p^{(n)} L(u_n(\tau)) \\ +N\sum_{n=0}^{\infty} p^{(n)} u_n(\tau)\end{array}\right) d\tau \quad (2.3)$$

$$-\int_0^t \lambda(\tau) g(\tau)\, d\tau$$

A comparison of similar powers of p gives solutions of various orders.

This method does not resort to linearization or assumptions of weak nonlinearity, the solutions generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

3.SOLUTION PROCEDURE

The Korteweg and Vries equation (KdV) equation has a form

$$\frac{\partial u}{\partial t} = \varepsilon u \frac{\partial u}{\partial x} - \mu \frac{\partial^3 u}{\partial x^3}$$

Where *u(x,t)* is the solution ,μ and ε are positive integer, *x* is the space variable and *t* is the time. For purpose of illustration of Variational Homotopy Perturbation method for solving the KdV equation $\varepsilon = 6$ and $\mu = 1$

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} \quad (3.1)$$

We start with an initial approximation

$$u(x,0) = -2\sec h^2 x$$

The exact solution of equation (3.1) in the closed form is

$$u(x,t) = -2\sec h^2 (x - 4t)$$

According to Variational Homotopy Perturbation method, we construct the correction functional for equation (3.1) as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \left[\lambda(\tau)\left(\frac{\partial u_n}{\partial t} - 6u_n \frac{\partial u_n}{\partial x} + \frac{\partial^3 u_n}{\partial x^3}\right)\right] d\tau$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\left(\frac{\partial u_n}{\partial t} - 6u_n \frac{\partial u_n}{\partial x} + \frac{\partial^3 u_n}{\partial x^3}\right)\right] d\tau$$

Applying the variational homotopy perturbation method and comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = -2\sec h^2 x$$

$$p^1 : u_1(x,t) = -16t \sec h^2 x \tanh x$$

$$p^2 : u_2(x,t) = -32t^2(-2 + \cosh 2x)\sec h^4 x$$

and so on .

Thus when we obtain the components the solution becomes

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 ....$$

4.CONVERGENT STUDY FOR THE SOLUTION

$$\begin{aligned} u(x,t) &= -2\sec h^2 x - 16t \sec h^2 x \tanh x \\ &\quad - 32t^2(-2 + \cosh 2x)\sec h^4 x \end{aligned} \quad (4.1)$$

For proving the convergence

$$-2 + \cosh 2x = -2 + \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

$$-2 + \cosh 2x = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 - 1$$

$$\begin{aligned} -2 + \cosh 2x &= \sinh^2 x - 1 \\ &= \cosh^2 x - 2 \end{aligned}$$

Thus the third approximation becomes

$$(-2 + \cosh 2x)\sec h^4 x$$

$$= (\cosh^2 x - 2)\frac{1}{\cosh^4 x}$$

$$= \sec h^2 x(1 - 2\sec h^2 x)$$

$$= \sec h^2 x\left[1 - \frac{1}{(e^x + e^{-x})^2}\right]$$

$= \sec h^2 x \tanh^2 x$

Thus equation (4.1) can be written as

$$u(x,t) = -2\sec h^2 x - 16t \sec h^2 x \tanh x - 32t^2 \sec h^2 x \tanh^2 x \qquad (4.2)$$

$$u(x,t) = -2\sec h^2 x \begin{bmatrix} 1 - 8t\tanh x - 16t^2 \tanh^2 x \\ +32t^3 \tanh^3 x + \ldots\ldots \end{bmatrix} \quad (4.3)$$

$$u(x,t) = -2\sec h^2 x \left[1 - \sum_{n=1}^{\infty} 2^{n+2} t^n \tanh^n x \right] \qquad (4.4)$$

Now for the equation (4.4)

We know that

$\left|\tanh x\right| \le 1$

And $0 < t < 1$

Thus $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$

Therefore the solution (4.1) is convergent by ratio test.

5.RESULTS AND DISCUSSIONS

In the present paper, an approximate solution is obtained of Equation (3.1) to emphasize the precision of the present method the numerical values are compared with the exact solution; results are summarized in Table 1 which shows that solutions are in excellent agreement with the numerical solutions and exact solutions at different values of x for specific values of *t*.

The 3-dimensional graphical representation of the solution of the KdV equation with space and time is shown in figure 1. In figure 2 , graph of exact solution versus x is given at different values of time.The graphs shows that solution are in good agreement with exact solution.Contour plot in figure 3 clearly shows waves.

From the three dimensional graph it clearly represents solitary wave observed by peak amplitude.

Table 1 Comparison of solution of KdV equation by VHPM at different times with the exact solution

	x	**Exact Solution**	**VHPM Solution**
t=0.01	-7.5	-0.00000225906	-0.00000225927090
	-2.5	-0.04914600344	-0.049150051640055
	2.5	-0.05754985288	-0.057545658380948
	7.5	-0.00000265103	-0.000002650825392
t=0.02	-7.5	-0.00000208538	-0.0000020869868
	-2.5	-0.04541063202	-0.0454424532021
	2.5	-0.06226782912	-0.06223366668393
	7.5	-0.00000287183	-0.00000287009587
t=0.03	-7.5	-0.00000192504	-0.00000193036505
	-2.5	-0.04195611960	-0.04206165805260
	2.5	-0.06736587306	-0.06724847827528
	7.5	-0.00000311102	-0.00000310502852
t=0.04	-7.5	-0.00000177704	-0.00000178940536
	-2.5	-0.03876179800	-0.03900766619142
	2.5	-0.07287344918	-0.07259009315499
	7.5	-0.00000337012	-0.00000335562333
t=0.05	-7.5	-0.00000164041	-0.00000166410784
	-2.5	-0.03580845352	-0.03628047761860
	2.5	-0.07882210802	-0.078258511323066
	7.5	-0.07882210802	-0.00000362188033

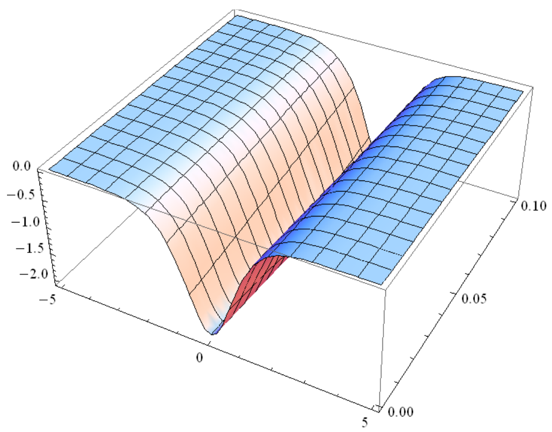

Fig 1 The behaviour solution of KdV equation versus x for different values of time by VHPM

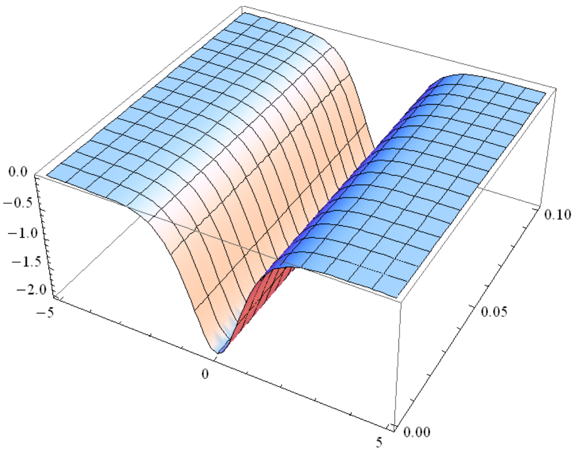

Fig 2 The behaviour of exact solution versus x for different values of time

A contour plot can also be plotted as follows However faster solitary wave will overtake the smaller and nonlinear part will be significant, after non linear interaction solitary waves retain their own identity suffering not more than a phase shift.

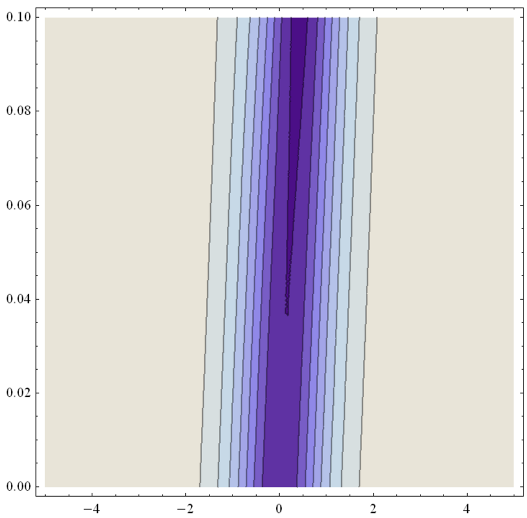

Figure 3 The contour plot of u(x, t) obtained by VHPM showing waves

6.CONCLUSION

The approximate analytical solution of KdV equation with specific initial conditions is obtained by Variational Homotopy Perturbation method in this paper . The results and compared with the exact solution available which is shown graphically. The main advantage of the algorithm is the method requires small size of computation compared to the other numerical methods. Also the solution obtained by variational homotopy perturbation method converges rapidly. Thus The VHPM can be applied for solving partial differential equation arising in different field of sciences.

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